Argument Filterings and Usable Rules
in Higher-Order Rewrite Systems

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The static dependency pair method is a method for proving the termination of higher-order rewrite systems *à la* Nipkow. It combines the dependency pair method introduced for first-order rewrite systems with the notion of strong computability introduced for typed λ-calculi. Argument filterings and usable rules are two important methods of the dependency pair framework used by current state-of-the-art first-order automated termination provers. In this presentation, we extend the class of higher-order systems on which the static dependency pair method can be applied. Then, we extend argument filterings and usable rules to higher-order rewriting, hence providing the basis for a powerful automated termination prover for higher-order rewrite systems.

1 Introduction

Various extensions of term rewriting systems (TRSs) [29] for handling functional variables and abstractions have been proposed [13, 22, 11, 23, 15]. In this paper, we consider higher-order rewrite systems (HRSs) [22], that is, rewriting on β-normal η-long simply-typed λ-terms using higher-order matching.

For example, the typical higher-order function *foldl* can be defined by the following HRS:

$$R_{\text{foldl}} = \left\{ \begin{array}{rcl} \text{foldl}(\lambda xy.F(x,y), X, \text{nil}) & \to & X \\ \text{foldl}(\lambda xy.F(x,y), X, \text{cons}(Y, L)) & \to & \text{foldl}(\lambda xy.F(x,y), F(X,Y), L) \end{array} \right.$$

Here we suppose that the function *foldl* has the type $(\mathbb{N} \to \mathbb{N} \to \mathbb{N}) \to L \to \mathbb{N}$, and L is a type of natural number's list. Then, the functions *sum* and *len*, computing the sum of the elements and the number of elements respectively, can be defined by the following HRSs:

$$\begin{array}{rcl} R_{\text{sum}} & = & R_{\text{foldl}} \cup \left\{ \begin{array}{rcl} \text{add}(0, Y) & \to & Y \\ \text{add}(\text{s}(X), Y) & \to & \text{s}(\text{add}(X, Y)) \\ \text{sum}(L) & \to & \text{foldl}(\lambda xy.\text{add}(x, y), 0, L) \end{array} \right. \\ R_{\text{len}} & = & R_{\text{foldl}} \cup \left\{ \begin{array}{rcl} \text{len}(L) & \to & \text{foldl}(\lambda xy.\text{s}(x), 0, L) \end{array} \right. \end{array}$$

In the HRS R_{len}, the anonymous function $\lambda xy.\text{s}(x)$ is represented by using λ-abstraction.

The static dependency pair method is a method for proving the termination of higher-order rewrite systems. It combines the dependency pair method introduced for first-order rewrite systems [1] with Tait and Girard's notion of strong computability introduced for typed λ-calculi [9]. It was first introduced for simply-typed term rewriting systems (STRSs) [17] and then extended to HRSs [19]. The static dependency pair method consists in showing the non-loopingness of each static recursion component independently, the set of static recursion components being computed through some static analysis of the possible sequences of function calls.

This method applies only to plain function-passing (PFP) systems. In this paper, we provide a new definition of PFP that significantly enlarges the class of systems on which the method can be applied. It is based on the notion of accessibility introduced in [3] and extended to HRSs in [2].

For the HRS $R_{\mathrm{sum}} \cup R_{\mathrm{len}}$, the static dependency pair method returns the following two components:

$$\{\ \mathrm{foldl}^\sharp(\lambda xy.F(x,y), X, \mathrm{cons}(Y, L)) \to \mathrm{foldl}^\sharp(\lambda xy.F(x,y), F(X,Y), L)\ \}$$
$$\{\ \mathrm{add}^\sharp(\mathrm{s}(X), Y) \to \mathrm{add}^\sharp(X, Y)\ \}$$

The static dependency pair method proves the termination of the HRS $R_{\mathrm{sum}} \cup R_{\mathrm{len}}$ by showing the non-loopingness of each component.

In order to show the non-loopingness of a component, the notion of reduction pair is often used. Roughly speaking, it consists in finding a well-founded quasi-ordering in which the component rules are strictly decreasing and all the original rules are non-increasing.

Argument filterings, which consist in removing some arguments of some functions, provide a way to generate reduction pairs. First introduced for TRSs [1], it has been extended to STRSs [15, 18]. In this paper, we extend it to HRSs.

In order to reduce the number of constraints required for showing the non-loopingness of a component, the notion of usable rules is also very important. Indeed, a finer analysis of sequences of function calls show that not all original rules need to be taken into account when trying to prove the termination of a component. This analysis was first conducted for TRSs [7, 10] and has been extended to STRSs [27, 18]. In this paper, we extend it to HRSs.

All together, this paper provides a strong theoretical basis for the development of an automated termination prover for HRSs, by extending to HRSs some successful techniques used by modern state-of-the-art first-order termination provers like for instance [8, 10].

The remainder of this paper is organized as follows. Section 2 introduces HRSs. Section 3 presents the static dependency pair method and extend the class of systems on which it can be applied. In Section 4, we extend the argument filtering method to HRSs. In Section 5, we extend the notion of usable rules on HRSs. Concluding remarks are given in Section 6.

2 Preliminaries

In this section, we introduce the basic notions for HRSs according to [22, 21].

The set $\mathcal{S}$ of *simple types* is generated from the set $\mathcal{B}$ of *basic types* by the type constructor $\to$. A *functional* or *higher-order type* is a simple type of the form $\alpha \to \beta$. We denote by $\rhd_s$ the strict subterm relation on types.

A *preterm* is generated from an infinite set of typed variables $\mathcal{V}$ and a set of typed function symbols Σ disjoint from $\mathcal{V}$ by λ-abstraction and λ-application. The set of typed preterms is denoted with $\mathcal{T}^{pre}$. We denote by $t\!\downarrow$ the η-long β-normal form of a simply-typed preterm t. The set $\mathcal{T}$ of *(simply-typed) terms* is defined as $\{t\!\downarrow \mid t \in \mathcal{T}^{pre}\}$. The unique type of a term t is denoted by $type(t)$. We write $\mathcal{V}_\alpha$ (resp. $\mathcal{T}_\alpha$) as the set of variables (resp. terms) of type α, The α-equivalence of terms is denoted by $\equiv$. The set of free variables in a term t is denoted by $FV(t)$. We assume for convenience that bound variables in a term are all different, and are disjoint from free variables. In general, a term t is of the form $\lambda x_1 \ldots x_m.a t_1 \ldots t_n$ where $a \in \Sigma \cup \mathcal{V}$. We abbreviate this by $\lambda \overline{x_m}.a(\overline{t_n})$. For a term $t \equiv \lambda \overline{x_m}.a(\overline{t_n})$, the symbol a, denoted by $top(t)$, is the *top symbol* of t, and the set $\{\overline{t_n}\}$, denoted by $args(t)$, is the *arguments* of t. We define the set $Sub(t)$ of *subterms* of t by $\{t\} \cup Sub(s)$ if $t \equiv \lambda x.s$, and $\{t\} \cup \bigcup_{i=1}^{n} Sub(t_i)$ if $t \equiv a(\overline{t_n})$. We use $t \unrhd_{sub} s$ to represent $s \in Sub(t)$, and define $t \rhd_{sub} s$ by $t \unrhd_{sub} s$ and $t \not\equiv s$. The set $Pos(t)$ of *positions* in a term t is the set of strings over positive integers inductively defined as $Pos(\lambda x.t) = \{\varepsilon\} \cup \{1p \mid p \in Pos(t)\}$ and $Pos(a(\overline{t_n})) = \{\varepsilon\} \cup \bigcup_{i=1}^{n}\{ip \mid p \in Pos(t_i)\}$. The *prefix order* $\prec$ on positions is defined by $p \prec q$ iff $pw = q$ for some $w \neq \varepsilon$. The subterm of t at position p is denoted by $t|_p$.

A term containing a unique occurrence of the special constant $\square_\alpha$ of type α is called a *context*, denoted by $C[\,]$. We use $C[t]$ for the term obtained from $C[\,]$ by replacing $\square_\alpha$ with $t \in \mathcal{T}_\alpha$. A substitution θ is a mapping from variables to terms such that $\theta(X)$ has the type of X for each variable X. We define $dom(\theta) = \{X \mid X\!\downarrow \not\equiv \theta(X)\}$ and assume that $dom(\theta)$ is always finite. A substitution θ is naturally extended to a mapping from terms to terms. We use $t\theta$ instead of $\theta(t)$ in the remainder of the paper. A substitution θ is said to be a *variable permutation* if $\forall X \in dom(\theta).\exists Y \in dom(\theta).\theta(X) \equiv Y\!\downarrow$ and $\theta(X) \equiv \theta(Y) \Rightarrow X = Y$ hold.

Following [21], a *higher-order rewrite rule* is a pair (l, r) of terms, denoted by $l \to r$, such that $top(l) \in \Sigma$, $type(l) = type(r) \in \mathcal{B}$ and $FV(l) \supseteq FV(r)$. Since, by definition, terms are in η-long form, function symbols are always applied to the same (maximal) number of arguments. Considering non-η-normal terms or rules of functional type is outside the scope of this paper. An HRS is a set of higher-order rewrite rules. The *reduction relation* $\underset{R}{\to}$ of an HRS R is defined by $s \underset{R}{\to} t$ iff $s \equiv C[l\theta\!\downarrow]$ and $t \equiv C[r\theta\!\downarrow]$ for some rewrite rule $l \to r \in R$, context $C[\,]$ and substitution θ. The transitive and reflexive-transitive closures of $\underset{R}{\to}$ are denoted by $\underset{R}{\overset{+}{\to}}$ and $\underset{R}{\overset{*}{\to}}$, respectively. An HRS R is said to be *finitely branching* if $\{t' \mid t \underset{R}{\to} t'\}$ is a finite set for any term t.

A term t is said to be *terminating* or *strongly normalizing* for an HRS R, denoted by $SN(R, t)$, if there is no infinite rewrite sequence of R starting from

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t. We write $SN(R)$ if $SN(R,t)$ holds for any term t. A well-founded relation $>$ on terms is a *reduction order* if $>$ is closed under substitution and context. We notice that an HRS R is terminating iff $R \subseteq >$ for some reduction order $>$.

A term t is said to be *strongly computable* in an HRS R if $SC(R,t)$ holds, which is inductively defined on simple types as follows: $SN(R,t)$ if $type(t) \in \mathcal{B}$, and $\forall u \in \mathcal{T}_\alpha.(SC(R,u) \Rightarrow SC(R,(tu)\downarrow))$ if $type(t) = \alpha \to \beta$. We also define the set $\mathcal{T}_{SC}^{args}(R) = \{t \mid \forall u \in args(t).SC(R,u)\}$.

Finally, we introduce the proposition required for later proof.

Proposition 2.1 *[21] If $s \xrightarrow[R]{*} t$ and $\theta \xrightarrow[R]{*} \theta'$ (i.e. $\forall x \in \mathcal{V}.x\theta \xrightarrow[R]{*} x\theta'$) then $s\theta\downarrow \xrightarrow[R]{*} t\theta'\downarrow$.*

3 Improved Static Dependency Pair Method

In this section, we introduce the static dependency pair method for plain function-passing (PFP) HRSs [19] but extend the class of PFP systems.

The method in [19] applies only to PFP systems. From a technical viewpoint, we have noticed that the unclosedness of strong computability with respect to the subterm relation is the reason why the method is not applicable to every HRS. Hence we can extend the applicable class for the method if more strongly computable subterms can be acquired. From the same motivation, Blanqui introduced the notion of accessibility to design a higher-order path ordering [2]. By using the notion of accessibility, we provide a new definition of PFP that enlarges the class of systems on which the method can be applied.

Definition 3.1 (Stable subterms) *The* stable subterms *of t are $SSub(t) = SSub_{FV(t)}(t)$ where $SSub_X(t) = \{t\} \cup SSub'_X(t)$, $SSub'_X(\lambda x.s) = SSub_X(s)$, $SSub'_X(a(\overline{t_n})) = \bigcup_{i=1}^n SSub_X(t_i)$ if $a \notin X$, and $SSub'_X(t) = \emptyset$ otherwise.*

Lemma 3.2 *(1) $SSub(t) \subseteq Sub(t)$. (2) If $u \in SSub(t)$ and $dom(\theta) \subseteq FV(t)$, then $u\theta\downarrow \in SSub(t\theta\downarrow)$. (3) If $u \in Sub(t)$ and $t \in SN$, then $u \in SN$.*

Definition 3.3 (Safe subterms - New definition) *The set of* safe subterms *of a term l is $safe(l) = \bigcup_{l' \in args(l)} \{t\downarrow \mid t \in Acc(l'), FV(t) \subseteq FV(l')\}$ where $t \in Acc(l')$ (t is* accessible *in l') if either:*

(0). $t = l'$,

(1). $t \in SSub(l')$, $type(t) \in \mathcal{B}$ and $FV(t) \subseteq FV(l')$,

(2). $\lambda x.t \in Acc(l')$ and $x \notin FV(l')$,

(3). $t(x\downarrow) \in Acc(l')$ and $x \notin FV(t) \cup FV(l')$,

(4). $f(\overline{t_n}) \in Acc(l')$, $t_i = \lambda\overline{x_k}.t$, $type(t) \in \mathcal{B}$ and $\{\overline{x_k}\} \cap FV(t) = \emptyset$,

(5). $x(\overline{t_n}) \in Acc(l')$, $t_i = t$ and $x \notin FV(\overline{t_n}) \cup FV(l')$.

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Strictly speaking, $safe(l)$ may not be included in $Sub(l)$ and, because of $((3))$, accessible terms are β-normal preterms not necessarily in η-long form.

Definition 3.4 (Plain Function-Passing [19]) *An HRS R is* plain function-passing *(PFP) if for any $l \to r \in R$ and $Z(\overline{r_n}) \in Sub(r)$ such that $Z \in FV(r)$, there exists $k \leq n$ such that $Z(\overline{r_k})\!\downarrow \,\in safe(l)$.*

For example, the HRS R_{foldl} displayed in the introduction is PFP, because $safe(\text{foldl}(\lambda xy.F(x,y), X, \text{cons}(Y,L))) = \{\lambda xy.F(x,y), X, \text{cons}(Y,L), Y, L\}$ and $F\!\downarrow \,\equiv \lambda xy.F(x,y) \in safe(\text{foldl}(\lambda xy.F(x,y), X, \text{cons}(Y,L)))$.

The definition of safeness given in [19] corresponds to cases $((0))$ and $((1))$. This new definition therefore includes much more terms, mainly higher-order patterns [20]. This greatly increases the class of rules that can be handled and the applicability of the method since it reduces the number of dependency pairs.

For instance, the new definition allows us to handle the following rule:

$$D(\lambda x.\sin(Fx))y \to D(\lambda x.Fx)y \times \cos(Fy)$$

Indeed, $l' = \lambda x.\sin(Fx) \in Acc(l')$ by $((0))$, $\sin(Fx) \in Acc(l')$ by $((2))$, $Fx \in Acc(l')$ by $((4))$ and $F \in Acc(l')$ by $((3))$. Therefore, $safe(l) = \{l', \lambda x.Fx, y\}$. With the previous definition, we had $safe(l) = \{l', y\}$ only.

Also, the new definition allows us to handle the following rule:

$$\forall(\lambda x.(Px \wedge Qx)) \to \forall(\lambda x.Px) \wedge \forall(\lambda x.Qx)$$

Indeed, $l' = \lambda x.(Px \wedge Qx) \in Acc(l')$ by $((0))$, $Px \wedge Qx \in Acc(l')$ by $((2))$, $Px, Qx \in Acc(l')$ by $((4))$, and $P, Q \in Acc(l')$ by $((3))$. Therefore, $safe(l) = \{l', \lambda x.Px, \lambda x.Qx\}$. With the previous definition, we had $safe(l) = \{l'\}$ only.

For the results presented in [19] to still hold, it suffices to check that this new definition of safeness still preserves strong computability (Lemma 4.3 in [19]). This can be shown by following the proof of Lemma 10 in [2].

Lemma 3.5 *Let R be an HRS and $l \to r \in R$. Then $l\theta\!\downarrow \,\in \mathcal{T}_{SC}^{args}(R)$ implies $SC(R, t\theta\!\downarrow)$ for any $t \in safe(l)$ and substitution θ.*

Proof. *We first prove that $t\theta\!\downarrow$ is strongly computable whenever $t \in Acc(l')$, $l'\theta\!\downarrow$ is strongly computable, and $x\theta$ is strongly computable for any $x \in FV(t)\backslash FV(l')$. Wlog we can assume that $dom(\theta) \subseteq FV(t)$. We prove the claim by induction on the definition of Acc.*

(0). Immediate.

(1). Since $l'\theta\!\downarrow$ is strongly computable, $l'\theta\!\downarrow$ is strongly normalizing. By Lemma 3.2, $t\theta\!\downarrow \,\in Sub(l'\theta\!\downarrow)$ and $t\theta\!\downarrow$ is SN. Therefore, since $type(t) \in \mathcal{B}$, $t\theta\!\downarrow$ is strongly computable.

(2). By definition of computability.

(3). *We have type*$(t) = \alpha \to \beta$. *So, let* $u \in \mathcal{T}_\alpha$ *strongly computable and* $\theta' = \theta \uplus \{x \mapsto u\}$ $(x \notin dom(\theta)$ *since* $x \notin FV(t))$. *Since* $x \notin FV(t)$, *we have* $(t\theta{\downarrow}u){\downarrow} = (t(x{\downarrow}))\theta'{\downarrow}$. *By IH,* $(t(x{\downarrow}))\theta'{\downarrow}$ *is strongly computable. Therefore,* $t\theta{\downarrow}$ *is strongly computable.*

(4). *Since strong computability on base types is equivalent to SN and* $\{\overline{x_k}\} \cap FV(t) = \emptyset$.

(5). *The term* $p_i = \lambda\overline{y_n}.y_i$ *can easily be proved strongly computable. Then, let* $\theta' = \theta \uplus \{x \mapsto p_i\}$ $(x \notin dom(\theta)$ *since* $x \notin FV(\overline{t_n}))$. *Since* $x \notin FV(t_i)$, *we have* $(x(\overline{t_n}))\theta'{\downarrow} = t_i\theta{\downarrow}$. *By induction hypothesis,* $(x(\overline{t_n}))\theta'{\downarrow}$ *is strongly computable. Therefore,* $t\theta{\downarrow} = t_i\theta{\downarrow}$ *is strongly computable.*

Let now $u \in safe(l)$. *We have* $u \equiv t{\downarrow}$ *for some* $t \in Acc(l')$ *and* $l' \in args(l)$ *with* $FV(t) \subseteq FV(l')$. *The term* $l'\theta{\downarrow}$ *is strongly computable since* $l\theta{\downarrow} \in \mathcal{T}_{SC}^{args}(R)$. *Since* $FV(t) \subseteq FV(l')$, *there is no* $x \in FV(t) \setminus FV(l')$. *Therefore,* $u\theta{\downarrow} \equiv t\theta{\downarrow}$ *is strongly computable.* □

This definition of safeness can be further improved (in case (4)) by using more complex interpretations for base types than just the set of strongly normalizing terms, but this requires to check more properties[5]. We leave this for future work.

We now recall the definitions of static dependency pair, static recursion component and reduction pair, and the basic theorems concerning these notions, including the subterm criterion [19].

Definition 3.6 (Static dependency pair [19]) *Let* R *be an HRS. All top symbols of the left-hand sides of rewrite rules, denoted by* $\mathcal{D}_R$, *are called* defined *symbols.*

We define the marked *term* $t^\sharp$ *by* $f^\sharp(\overline{t_n})$ *if* t *has the form* $f(\overline{t_n})$ *with* $f \in \mathcal{D}_R$; *otherwise* $t^\sharp \equiv t$. *Then, let* $\mathcal{D}_R^\sharp = \{f^\sharp \mid f \in \mathcal{D}_R\}$.

We also define the set of candidate subterms *as follows:* $Cand(\lambda\overline{x_m}.a(\overline{t_n})) = \{\lambda\overline{x_m}.a(\overline{t_n})\} \cup \bigcup_{i=1}^n Cand(\lambda\overline{x_m}.t_i)$.

Now, a pair $\langle l^\sharp, a^\sharp(\overline{r_n}) \rangle$, *denoted by* $l^\sharp \to a^\sharp(\overline{r_n})$, *is said to be a* static dependency pair *in* R *if there exists* $l \to r \in R$ *such that* $\lambda\overline{x_m}.a(\overline{r_n}) \in Cand(r)$, $a \in \mathcal{D}_R$, *and* $a(\overline{r_k}){\downarrow} \notin safe(l)$ *for all* $k \leq n$. *We denote by* $SDP(R)$ *the set of static dependency pairs in* R.

Example 3.7 *Let* R_{ave} *be the following PFP-HRS:*

$$
R_{\text{ave}} = R_{\text{sum}} \cup R_{\text{len}} \cup \left\{
\begin{array}{rcl}
\text{sub}(X, 0) & \to & X \\
\text{sub}(0, Y) & \to & 0 \\
\text{sub}(\text{s}(X), \text{s}(Y)) & \to & \text{sub}(X, Y) \\
\text{div}(0, \text{s}(Y)) & \to & 0 \\
\text{div}(\text{s}(X), \text{s}(Y)) & \to & \text{s}(\text{div}(\text{sub}(X, Y), \text{s}(Y))) \\
\text{ave}(L) & \to & \text{div}(\text{sum}(L), \text{len}(L))
\end{array}
\right.
$$

Then, the set $SDP(R_{\text{ave}})$ consists of the following eleven pairs:

$$
\left\{
\begin{array}{rcl}
\text{foldl}^\sharp(\lambda xy.F(x,y), X, \text{cons}(Y,L)) & \to & \text{foldl}^\sharp(\lambda xy.F(x,y), F(X,Y), L) \\
\text{add}^\sharp(\text{s}(X), Y) & \to & \text{add}^\sharp(X, Y) \\
\text{sum}^\sharp(L) & \to & \text{foldl}^\sharp(\lambda xy.\text{add}(x,y), 0, L) \\
\text{sum}^\sharp(L) & \to & \text{add}^\sharp(x, y) \\
\text{sub}^\sharp(\text{s}(X), \text{s}(Y)) & \to & \text{sub}^\sharp(X, Y) \\
\text{div}^\sharp(\text{s}(X), \text{s}(Y)) & \to & \text{div}^\sharp(\text{sub}(X,Y), \text{s}(Y)) \\
\text{div}^\sharp(\text{s}(X), \text{s}(Y)) & \to & \text{sub}^\sharp(X, Y) \\
\text{len}^\sharp(L) & \to & \text{foldl}^\sharp(\lambda xy.\text{s}(x), 0, L) \\
\text{ave}^\sharp(L) & \to & \text{div}^\sharp(\text{sum}(L), \text{len}(L)) \\
\text{ave}^\sharp(L) & \to & \text{sum}^\sharp(L) \\
\text{ave}^\sharp(L) & \to & \text{len}^\sharp(L)
\end{array}
\right.
$$

Definition 3.8 (Static dependency chain [19]) *Let R be an HRS. A sequence $u_0^\sharp \to v_0^\sharp, u_1^\sharp \to v_1^\sharp, \ldots$ of static dependency pairs is a* static dependency chain *in R if there exist $\theta_0, \theta_1, \ldots$ such that $v_i^\sharp \theta_i \downarrow \xrightarrow{*}_R u_{i+1}^\sharp \theta_{i+1} \downarrow$ and $u_i \theta_i \downarrow, v_i \theta_i \downarrow \in \mathcal{T}_{SC}^{args}(R)$ for all i.*

Note that, for all i, $u_i^\sharp \theta_i$ and $v_i^\sharp \theta_i$ are terminating, since strong computability implies termination.

Proposition 3.9 *[19] Let R be a PFP-HRS. If there exists no infinite static dependency chain then R is terminating.*

Proof. *By using Lemma 3.5 instead of Lemma 4.3 in [19], the proof of the correspondence theorem (Theorem 5.23 in [19]) still holds.* □

Definition 3.10 (Static recursion component [19]) *Let R be an HRS. The* static dependency graph *of R is the directed graph in which nodes are $SDP(R)$ and there exists an arc from $u^\sharp \to v^\sharp$ to $u'^\sharp \to v'^\sharp$ if the sequence $u^\sharp \to v^\sharp$, $u'^\sharp \to v'^\sharp$ is a static dependency chain.*

A static recursion component *is a set of nodes in a strongly connected subgraph of the static dependency graph of R. We denote by $SRC(R)$ the set of static recursion components of R.*

A static recursion component C is non-looping *if there exists no infinite static dependency chain in which only pairs in C occur and every $u^\sharp \to v^\sharp \in C$ occurs infinitely many times.*

Proposition 3.11 *[19] Let R be a PFP-HRS such that there exists no infinite path in the static dependency graph. If all static recursion components are non-looping, then R is terminating.*
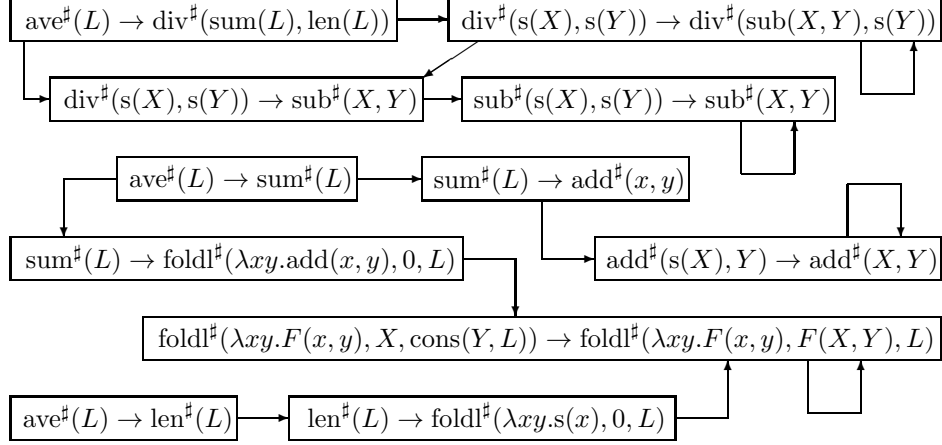
Figure 1: The static dependency graph of R_{ave}

Example 3.12 *For the PFP-HRS R_{ave} in Example 3.7, the static dependency graph of R_{ave} is shown in Fig. 1. Then the set $SRC(R_{\text{ave}})$ consists of the following four static recursion components:*

$$\{ \ \text{foldl}^\sharp(\lambda xy.F(x,y), X, \text{cons}(Y,L)) \to \text{foldl}^\sharp(\lambda xy.F(x,y), F(X,Y), L) \ \}$$
$$\{ \ \text{add}^\sharp(\text{s}(X),Y) \to \text{add}^\sharp(X,Y) \ \}$$
$$\{ \ \text{sub}^\sharp(\text{s}(X),\text{s}(Y)) \to \text{sub}^\sharp(X,Y) \ \}$$
$$\{ \ \text{div}^\sharp(\text{s}(X),\text{s}(Y)) \to \text{div}^\sharp(\text{sub}(X,Y),\text{s}(Y)) \ \}$$

In order to prove the non-loopingness of components, the notions of subterm criterion and reduction pair have been proposed. The subterm criterion was introduced on TRSs [10], and then extended to STRSs [17] and HRSs [19]. Reduction pairs [16] are an abstraction of the notion of weak-reduction order [1].

Definition 3.13 (Subterm criterion [19]) *Let R be an HRS and $C \in SRC(R)$. We say that C satisfies the* subterm criterion *if there exists a function π from $\mathcal{D}_R^\sharp$ to non-empty sequences of positive integers such that:*

- $u|_{\pi(top(u^\sharp))} \rhd_{sub} v|_{\pi(top(v^\sharp))}$ *for some $u^\sharp \to v^\sharp \in C$,*

- *and the following conditions hold for every $u^\sharp \to v^\sharp \in C$:*

    - $u|_{\pi(top(u^\sharp))} \unrhd_{sub} v|_{\pi(top(v^\sharp))}$,
    - $\forall p \prec \pi(top(u^\sharp)).top(u|_p) \notin FV(u)$,
    - *and $\forall q \prec \pi(top(v^\sharp)).q = \varepsilon \vee top(v|_q) \notin FV(v) \cup \mathcal{D}_R$.*

Definition 3.14 (Reduction pair, Weak reduction order [1, 16]) *A pair $(\gtrsim, >)$ of relations is a* reduction pair *if $\gtrsim$ and $>$ satisfy the following properties:*

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- $>$ is well-founded and closed under substitutions,

- $\gtrsim$ is closed under contexts and substitutions,

- and $\gtrsim \cdot > \,\subseteq\, >$ or $> \cdot \gtrsim \,\subseteq\, >$.

In particular, $\gtrsim$ is a weak reduction order if $(\gtrsim, \gtrsim \,\setminus\, \lesssim)$ is a reduction pair.

Proposition 3.15 *[19] Let R be a PFP-HRS such that there exists no infinite path in the static dependency graph. Then, $C \in SRC(R)$ is non-looping if C satisfies one of the following properties:*

- *C satisfies the subterm criterion.*

- *There is a reduction pair $(\gtrsim, >)$ such that $R \subseteq \gtrsim$, $C \subseteq \gtrsim \cup >$ and $C \cap > \,\neq\, \emptyset$.*

Example 3.16 *Let $\pi(\mathrm{foldl}^\sharp) = 3$ and $\pi(\mathrm{add}^\sharp) = \pi(\mathrm{sub}^\sharp) = 1$. Then, every static recursion component C except the one for* div *(cf. Example 3.12) satisfies the subterm criterion in the underlined positions below. Hence, these static recursion components are non-looping.*

$$\left\{\mathrm{foldl}^\sharp(\lambda xy.F(x,y), X, \underline{\mathrm{cons}(Y,L)}) \to \mathrm{foldl}^\sharp(\lambda xy.F(x,y), F(X,Y), \underline{L})\right\}$$
$$\left\{\mathrm{add}^\sharp(\underline{\mathrm{s}(X)}, Y) \to \mathrm{add}^\sharp(\underline{X}, Y)\right\} \quad \left\{\mathrm{sub}^\sharp(\underline{\mathrm{s}(X)}, \mathrm{s}(Y)) \to \mathrm{sub}^\sharp(\underline{X}, Y)\right\}$$

4 Argument Filterings

An argument filtering generates a weak reduction order from an arbitrary reduction order. The method was first proposed on TRSs [1], and then extended to STRSs [15, 18]. Since this extension has the problem that this method may destroy the well-typedness of terms, Kusakari and Sakai improved the method so that the well-typedness is never destroyed [18]. In this section, we expand this technique to HRSs.

Definition 4.1 *An argument filtering function is a function π such that, for every $f \in \Sigma$ of type $\alpha_1 \to \cdots \to \alpha_n \to \beta$ with $\beta \in \mathcal{B}$, $\pi(f)$ is either a positive integer $i \leq n$ if $\alpha_i = \beta$, or a list of positive integers $[i_1, \ldots, i_k]$ with $i_1, \ldots, i_k \leq n$. Then, we extend the function π to terms by taking:*

$$\pi(\lambda\overline{x_m}.a(\overline{t_n})) \equiv \begin{cases} \lambda\overline{x_m}.\pi(t_i) & \text{if } a \in \Sigma \text{ and } \pi(a) = i \\ \lambda\overline{x_m}.a(\pi(t_{i_1}), \ldots, \pi(t_{i_k})) & \text{if } a \in \Sigma \text{ and } \pi(a) = [i_1, \ldots, i_k] \\ \lambda\overline{x_m}.a(\pi(t_1), \ldots, \pi(t_n)) & \text{if } a \in \mathcal{V} \end{cases}$$

Given an argument filtering π and a binary relation $>$, we define $s \gtrsim_\pi t$ by $\pi(s) > \pi(t)$ or $\pi(s) \equiv \pi(t)$, and $s >_\pi t$ by $\pi(s) > \pi(t)$. We also define

the substitution θ_π by $\theta_\pi(x) \equiv \pi(\theta(x))$. Finally, we define the typing function $type_\pi$ after argument filtering as $type_\pi(a) = \alpha_{i_1} \to \cdots \to \alpha_{i_k} \to \beta$ if $a \in \Sigma$, $\pi(a) = [i_1, \ldots, i_k]$, $type(a) = \alpha_1 \to \cdots \alpha_n \to \beta$ and $\beta \in \mathcal{B}$; otherwise $type_\pi(a) = type(a)$.

In the examples, except stated otherwise, $\pi(f) = [1, \ldots, n]$ if $type(f) = \alpha_1 \to \cdots \to \alpha_n \to \beta$ and $\beta \in \mathcal{B}$ (no argument is removed).

For instance, if $\pi(\mathrm{sub}) = [1]$ then $\pi(\mathrm{div}^\sharp(\mathrm{sub}(X, Y), \mathrm{s}(Y))) \equiv \mathrm{div}^\sharp(\mathrm{sub}(X), \mathrm{s}(Y))$.

Note that our argument filtering method never destroys the well-typedness, which is easily proved by induction on terms.

Theorem 4.2 *For any argument filtering π and term $t \in \mathcal{T}$, $\pi(t)$ is well-typed under the typing function $type_\pi$ and $type_\pi(\pi(t)) = type(t)$.*

In the following, we prove the soundness of the argument filtering method as a generating method of weak reduction orders. To this end, we first prove a lemma required for showing that $>_\pi$ and $\gtrsim_\pi$ are closed under substitution.

Lemma 4.3 $\pi(t\theta{\downarrow}) \equiv \pi(t)\theta_\pi{\downarrow}$.

Proof. *We proceed by induction on preterm $t\theta$ ordered with $\underset{\beta'}{\to} \cup \triangleright_{sub}$.*

- *In case of $t \equiv \lambda x.u$: Since $t\theta \triangleright_{sub} u\theta$, we have $\pi(u\theta{\downarrow}) \equiv \pi(u)\theta_\pi{\downarrow}$ from the induction hypothesis. Hence we have: $\pi((\lambda x.u)\theta{\downarrow}) \equiv \lambda x.\pi(u\theta{\downarrow}) \equiv \lambda x.\pi(u)\theta_\pi{\downarrow} \equiv \pi(\lambda x.u)\theta_\pi{\downarrow}$.*

- *In case of $t \equiv f(\overline{t_n})$, $f \in \Sigma$, and $\pi(f) = i$: Since $t\theta \triangleright_{sub} t_i\theta$, we have $\forall i.\, \pi(t_i\theta{\downarrow}) \equiv \pi(t_i)\theta_\pi{\downarrow}$ from the induction hypothesis. Hence we have: $\pi(f(\overline{t_n})\theta{\downarrow}) \equiv \pi(f(\overline{t_n\theta{\downarrow}})) \equiv \pi(t_i\theta{\downarrow}) \equiv \pi(t_i)\theta_\pi{\downarrow} \equiv \pi(f(\overline{t_n}))\theta_\pi{\downarrow}$.*

- *In case of $t \equiv f(\overline{t_n})$, $f \in \Sigma$, and $\pi(f)$ is a list: Suppose that $t'_i \equiv \bot{\downarrow}$ if $i \notin \pi(f)$; otherwise $t'_i \equiv \pi(t_i)$, and $t''_i \equiv \bot{\downarrow}$ if $i \notin \pi(f)$; otherwise $t''_i \equiv \pi(t_i\theta{\downarrow})$. For each i, since $t\theta \triangleright_{sub} t_i\theta$, we have $\pi(t_i\theta{\downarrow}) \equiv \pi(t_i)\theta_\pi{\downarrow}$ from the induction hypothesis. Then $t''_i \equiv t'_i\theta_\pi{\downarrow}$ holds for each i. Hence we have: $\pi(f(\overline{t_n})\theta{\downarrow}) \equiv \pi(f(\overline{t_n\theta{\downarrow}})) \equiv f(\overline{t''_n}) \equiv f(\overline{t'_n\theta_\pi{\downarrow}}) \equiv f(\overline{t'_n})\theta_\pi{\downarrow} \equiv \pi(f(\overline{t_n}))\theta_\pi{\downarrow}$.*

- *In case of $t \equiv X \in \mathcal{V}$: Obvious from the definition of θ_π.*

- *In case of $t \equiv X(\overline{t_n})$, $X \in \mathcal{V}$ and $n > 0$: Since $type(X) = type(X\theta)$, we have $X\theta \equiv \lambda\overline{y_n}.a(\overline{u_k})$. For each i, since $t\theta \triangleright_{sub} t_i\theta$, we have $\pi(t_i\theta{\downarrow}) \equiv \pi(t_i)\theta_\pi{\downarrow}$ from the induction hypothesis. Since $t\theta \equiv (\lambda\overline{y_n}.a(\overline{u_k}))(\overline{t_n\theta}) \overset{+}{\underset{\beta}{\to}} a(\overline{u_k})\{\, y_i := t_i\theta{\downarrow} \mid i \in \overline{n}\,\}$, we have $\pi(a(\overline{u_k})\{y_i := t_i\theta{\downarrow} \mid i \in \overline{n}\}{\downarrow}) \equiv \pi(a(\overline{u_k}))\{y_i := \pi(t_i\theta{\downarrow}) \mid i \in \overline{n}\}{\downarrow}$ from the induction hypothesis. Hence we have: $\pi(X(\overline{t_n})\theta{\downarrow}) \equiv \pi((\lambda\overline{y_n}.a(\overline{u_k}))(\overline{t_n\theta{\downarrow}}){\downarrow}) \equiv \pi(a(\overline{u_k})\{\, y_i := t_i\theta{\downarrow} \mid i \in \overline{n}\,\}{\downarrow}) \equiv \pi(a(\overline{u_k}))\{\, y_i := \pi(t_i\theta{\downarrow}) \mid i \in \overline{n}\,\}{\downarrow} \equiv \pi(a(\overline{u_k}))\{\, y_i := \pi(t_i)\theta_\pi{\downarrow} \mid i \in \overline{n}\,\}{\downarrow} \equiv (\lambda\overline{y_n}.\pi(a(\overline{u_k})))(\overline{\pi(t_n)\theta_\pi{\downarrow}}){\downarrow} \equiv \pi(\lambda\overline{y_n}.a(\overline{u_k}))(\overline{\pi(t_n)\theta_\pi{\downarrow}}){\downarrow} \equiv X(\overline{\pi(t_n)})\theta_\pi{\downarrow} \equiv \pi(X(\overline{t_n}))\theta_\pi{\downarrow}$.* $\quad\square$

Note that the corresponding lemma in STRSs is $\pi(t\theta) \geq \pi(t)\theta_\pi$ where $>$ is a given binary relation [18]. This is the technical reason why the argument filtering method on STRSs can apply to only left-firmness (left-hand side variables occurs at leaf positions only) STRSs[15, 18]. This difference originates the fact that STRSs allow partial application (ex. foldl F, foldl F X) but HRSs does not.

Theorem 4.4 *For any reduction order $>$ and argument filtering function π, $\gtrsim_\pi$ is a weak reduction order.*

Proof. *It is easily shown that $s \gtrsim_\pi t \Rightarrow C[s] \gtrsim_\pi C[t]$ by induction on $C[\,]$. From Lemma 4.3, we have $s \gtrsim_\pi t \Rightarrow \pi(s) \geq \pi(t) \Rightarrow \pi(s)\theta_\pi\downarrow \geq \pi(t)\theta_\pi\downarrow \Rightarrow \pi(s\theta\downarrow) \geq \pi(t\theta\downarrow) \Rightarrow s\theta\downarrow \gtrsim_\pi t\theta\downarrow$, and $s >_\pi t \Rightarrow \pi(s) > \pi(t) \Rightarrow \pi(s)\theta_\pi\downarrow > \pi(t)\theta_\pi\downarrow \Rightarrow \pi(s\theta\downarrow) > \pi(t\theta\downarrow) \Rightarrow s\theta\downarrow >_\pi t\theta\downarrow$. Remaining properties are routine.* $\qquad\square$

Example 4.5 *Consider the PFP-HRS R_{ave} in Example 3.7. Every static recursion component except $\{\mathrm{div}^\sharp(\mathrm{s}(X), \mathrm{s}(Y)) \to \mathrm{div}^\sharp(\mathrm{sub}(X, Y), \mathrm{s}(Y))\}$ is non-looping (cf. Example 3.16). We can prove its non-loopingness with the argument filtering method, by taking $\pi(\mathrm{sub}) = \pi(\mathrm{div}^\sharp) = [1]$, and the normal higher-order reduction ordering $>^n_{rhorpo}$, written $(>_{rhorpo})_n$ in [12] defined by:*

- *a neutralization level $\mathcal{L}^j_f = 0$ for all symbol $f \in \Sigma$ and argument position j (in fact, these parameters are relevant for functional arguments only),*

- *filtering out all arguments (a notion introduced in [12] not to be confused with the argument filtering method) by taking $\mathcal{A}^j_f = \emptyset$ for all f and j (again, these parameters are relevant for functional arguments only),*

- *a precedence $s_{new} >_{\Sigma_{new}} sub_{new}$ (a symbol f_{new} with $f \in \Sigma$ is a new symbol introduced by the definition of $>^n_{rhorpo}$ in [12], with the same type as f since neutralization levels are null),*

- *a multiset (or lexicographic) status for $\mathrm{div}^\sharp_{new}$,*

- *a quasi-ordering on types reduced to the equality (the strict part is well-founded since it is empty, and equality preserves functional types).*

Then we have $\pi(\mathrm{div}^\sharp(\mathrm{s}(X), \mathrm{s}(Y))) \equiv \mathrm{div}^\sharp(\mathrm{s}(X)) >^n_{rhorpo} \mathrm{div}^\sharp(\mathrm{sub}(X)) \equiv \pi(\mathrm{div}^\sharp(\mathrm{sub}(X, Y), \mathrm{s}(Y)))$, and $R_{\mathrm{div}} \subseteq (\geq^n_{rhorpo})_\pi$. For instance, $\mathrm{div}^\sharp(\mathrm{s}(X)) >^n_{rhorpo} \mathrm{div}^\sharp(\mathrm{sub}(X))$ since $FN(\mathrm{div}^\sharp(\mathrm{s}(X)))\downarrow_\beta >_{rhorpo} FN(\mathrm{div}^\sharp(\mathrm{sub}(X)))\downarrow_\beta$ and, because $\mathcal{L}^j_f = 0$ and $\mathcal{A}^j_f = \emptyset$, $FN(ft_1 \ldots t_n) = f_{new} FN(t_1) \ldots FN(t_n)$. From Proposition 3.15, the static recursion component for div is non-looping, and R_{div} is terminating.

5 Usable Rules

In order to reduce the number of constraints required for showing the non-loopingness of a component, the notion of usable rules is widely used. This notion was introduced on TRSs [7, 10] and then extended to STRSs [27, 18]. In this section, we extend it to HRSs.

To illustrate the interest of this notion, we start with some example.

Example 5.1 *We consider the data type* heap ::= leaf | node(nat, heap, heap) *and the PFP-HRS R_{heap} defined by the following rules:*

$$
\left\{
\begin{array}{rcl}
\text{add}(0, Y) & \to & Y \\
\text{add}(\text{s}(X), Y) & \to & \text{s}(\text{add}(X, Y)) \\
\text{map}(\lambda x.F(x), \text{nil}) & \to & \text{nil} \\
\text{map}(\lambda x.F(x), \text{cons}(X, L)) & \to & \text{cons}(F(X), \text{map}(\lambda x.F(x).L)) \\
\text{merge}(H, \text{leaf}) & \to & H \\
\text{merge}(\text{leaf}, H) & \to & H \\
\multicolumn{3}{l}{\text{merge}(\text{node}(X_1, H_{11}, H_{12}), \text{node}(X_2, H_{21}, H_{22}))} \\
\multicolumn{3}{r}{\to \text{node}(X_1, H_{11}, \text{merge}(H_{12}, \text{node}(X_2, H_{21}, H_{22}))} \\
\multicolumn{3}{l}{\text{merge}(\text{node}(X_1, H_{11}, H_{12}), \text{node}(X_2, H_{21}, H_{22}))} \\
\multicolumn{3}{r}{\to \text{node}(X_2, \text{merge}(\text{node}(X_1, H_{11}, H_{12}), H_{21}), H_{22})} \\
\text{foldT}(\lambda xyz.F(x, y, z), X, \text{leaf}) & \to & X \\
\multicolumn{3}{l}{\text{foldT}(\lambda xyz.F(x, y, z), X, \text{node}(Y, H_1, H_2))} \\
\multicolumn{3}{r}{\to F(X, \text{foldT}(\lambda xyz.F(x, y, z), X, H_1), \text{foldT}(\lambda xyz.F(x, y, z), X, H_2))} \\
\text{sumT}(H) & \to & \text{foldT}(\lambda xyz.\text{add}(x, \text{add}(y, z)), 0, H) \\
\text{hd}(\text{nil}) & \to & \text{leaf} \\
\text{hd}(\text{cons}(X, L)) & \to & X \\
\text{l2t}(\text{nil}) & \to & \text{nil} \\
\text{l2t}(\text{cons}(H, \text{nil})) & \to & \text{cons}(H, \text{nil}) \\
\text{l2t}(\text{cons}(H_1, \text{cons}(H_2, L))) & \to & \text{l2t}(\text{cons}(\text{merge}(H_1, H_2), \text{l2t}(L))) \\
\text{list2heap}(L) & \to & \text{hd}(\text{l2t}(\text{map}(\lambda x.\text{node}(x, \text{leaf}, \text{leaf}), L)))
\end{array}
\right.
$$

The static recursion components for foldT *consists of*

$$\{\text{foldT}^\sharp(\lambda xyz.F(x, y, z), X, \text{node}(Y, H_1, H_2)) \to \text{foldT}(\lambda xyz.F(x, y, z), X, H_i)\}$$

for $i = 1, 2$, and their union. By taking $\pi(\text{foldT}) = 3$, these components satisfy the subterm criterion. The static recursion components for add, map *and* merge *also satisfy the subterm criterion. Hence it suffices to show that the following three static recursion components for* l2t *are non-looping:*

$$\{ \text{l2t}^\sharp(\text{cons}(H_1, \text{cons}(H_2, L))) \to \text{l2t}^\sharp(\text{cons}(\text{merge}(H_1, H_2), \text{l2t}(L))) \cdots (1) \}$$
$$\{ \text{l2t}^\sharp(\text{cons}(H_1, \text{cons}(H_2, L))) \to \text{l2t}^\sharp(L) \cdots (2) \}$$
$$\{(1), (2)\}$$

The component $\{(2)\}$ satisfies the subterm criterion. By taking $\pi(\text{cons}) = [2]$ and $\pi(\text{l2t}) = \pi(\text{l2t}^\sharp) = 1$, we can orient the static dependency pairs (1) and (2)

by using the normal higher-order recursive path ordering [12]:

$$\pi(\mathrm{l2t}^\sharp(\mathrm{cons}(H_1, \mathrm{cons}(H_2, L))))$$
$$\equiv \mathrm{cons}(\mathrm{cons}(L)) >^n_{rhorpo} \mathrm{cons}(L) \equiv \pi(\mathrm{l2t}^\sharp(\mathrm{cons}(\mathrm{merge}(H_1, H_2), \mathrm{l2t}(L))))$$
$$\pi(\mathrm{l2t}^\sharp(\mathrm{cons}(H_1, \mathrm{cons}(H_2, L)))) \equiv \mathrm{cons}(\mathrm{cons}(L)) >^n_{rhorpo} L \equiv \pi(\mathrm{l2t}^\sharp(L))$$

However, in contrast to Example 4.5, the non-loopingness of $\{(1)\}$ and $\{(1), (2)\}$ cannot be shown with the previous techniques. Indeed, we cannot solve the constraint $R_{\mathrm{heap}} \subseteq \gtrsim$. More precisely, we cannot orient the rule for hd*, because $\pi(\mathrm{hd}(\mathrm{cons}(X, L))) \equiv \mathrm{hd}(\mathrm{cons}(L))$ does not contain the variable X occurring in the right-hand side.*

The notion of usable rule solves this problem, that is, it allows us to ignore the rewrite rule for hd for showing the non-loopingness of l2t.

Definition 5.2 (Usable rules) *We denote $f >_{\mathrm{def}} g$ if g is a defined symbol and there is some $l \to r \in R$ such that $top(l) = f$ and g occurs in r.*

We define the set $\mathcal{U}(t)$ of usable rules of a term t as follows. If, for every $X(\overline{t_n}) \in Sub(t)$, $\overline{t_n}$ are distinct bound variables, then $\mathcal{U}(t) = \{l \to r \in R \mid f >^*_{\mathrm{def}} top(l)$ for some $f \in \mathcal{D}_R$ occurs in $t\}$. Otherwise, $\mathcal{U}(t) = R$. The usable rules of a static recursion component C is $\mathcal{U}(C) = \bigcup\{\mathcal{U}(v^\sharp) \mid u^\sharp \to v^\sharp \in C\}$.

For each $\alpha \in \mathcal{B}$, we associate the new function symbols $\perp_\alpha$ and c_α with $type(\perp_\alpha) = \alpha$ and $type(\mathrm{c}_\alpha) = \alpha \to \alpha \to \alpha$. We define the HRS C_e as $C_e = \{\mathrm{c}_\alpha(x_1, x_2) \to x_i \mid \alpha \in \mathcal{B}, i = 1, 2\}$.

Hereafter we omit the index α whenever no confusion arises.

When we show the non-loopingness of a static recursion component using a reduction pair, Proposition 3.15 requires showing that $R \subseteq \gtrsim$. The non-loopingness is not guaranteed by simply replacing R with $\mathcal{U}(C)$. We can supplement the gap with the HRS C_e.

Theorem 5.3 *Let R be a finitely-branching PFP-HRS. Then $C \in SRC(R)$ is non-looping if there exists a reduction pair $(\gtrsim, >)$ such that $\mathcal{U}(C) \cup C_e \subseteq \gtrsim$, $C \subseteq \gtrsim \cup >$, and $C \cap > \neq \emptyset$.*

The proof of this theorem will be given at the end of this section.

Example 5.4 *We show the termination of the PFP-HRS R_{heap} in Example 5.1. We have to show the non-loopingness of the components $\{(1)\}$ and $\{(1), (2)\}$. To this end, it suffices to show that the constraint $\mathcal{U}(\{(1), (2)\}) \cup C_e \subseteq \gtrsim$ can be solved (instead of $R_{\mathrm{heap}} \subseteq \gtrsim$). The usable rules of $\{(1), (2)\}$ are:*

$$\left\{ \begin{array}{rcl}
\mathrm{merge}(H, \mathrm{leaf}) & \to & H \\
\mathrm{merge}(\mathrm{leaf}, H) & \to & H \\
\mathrm{merge}(\mathrm{node}(X_1, H_{11}, H_{12}), \mathrm{node}(X_2, H_{21}, H_{22})) & & \\
& \multicolumn{2}{l}{\to \mathrm{node}(X_1, H_{11}, \mathrm{merge}(H_{12}, \mathrm{node}(X_2, H_{21}, H_{22})))} \\
\mathrm{merge}(\mathrm{node}(X_1, H_{11}, H_{12}), \mathrm{node}(X_2, H_{21}, H_{22})) & & \\
& \multicolumn{2}{l}{\to \mathrm{node}(X_2, \mathrm{merge}(\mathrm{node}(X_1, H_{11}, H_{12}), H_{21}), H_{22})} \\
\mathrm{l2t}(\mathrm{nil}) & \to & \mathrm{nil} \\
\mathrm{l2t}(\mathrm{cons}(H, \mathrm{nil})) & \to & \mathrm{cons}(H, \mathrm{nil}) \\
\mathrm{l2t}(\mathrm{cons}(H_1, \mathrm{cons}(H_2, L))) & \to & \mathrm{l2t}(\mathrm{cons}(\mathrm{merge}(H_1, H_2), \mathrm{l2t}(L)))
\end{array} \right.$$

The weak reduction order $(>^n_{rhorpo})_\pi$ orient the rules. Since $C_e \subseteq (>^n_{rhorpo})_\pi$, we conclude that R_{heap} is terminating.

In the rest of this section, we present a proof of Theorem 5.3. We assume that R is a finitely-branching PFP-HRS, C is a static recursion component of R, and $\Delta = \{top(l) \mid l \to r \in R \setminus \mathcal{U}(C)\}$.

The key idea of the proof is to use the following interpretation I.

Thanks to the Well-ordering theorem, we assume that every non-empty set of terms T has a least element $least(T)$.

Definition 5.5 *For a terminating term $t \in \mathcal{T}_\alpha$, $I(t)$ is defined as follows:*

$$
I(t) \equiv \begin{cases}
\lambda x.I(t') & \text{if } t \equiv \lambda x.t' \\
a(\overline{I(t_n)}) & \text{if } t \equiv a(\overline{t_n}) \text{ and } a \notin \Delta \\
c_\alpha(a(\overline{I(t_n)}), Red_\alpha(\{I(t') \mid t \xrightarrow[R \setminus \mathcal{U}(C)]{} t'\})) & \text{if } t \equiv a(\overline{t_n}) \text{ and } a \in \Delta
\end{cases}
$$

Here, for each $\alpha \in \mathcal{B}$, $Red_\alpha(T)$ is defined as $\perp_\alpha$ if $T = \emptyset$; otherwise $c_\alpha(u, Red_\alpha(T \setminus \{u\}))$ where $u \equiv least(T)$. We also define θ^I by $\theta^I(x) \equiv I(\theta(x))$ for a terminating substitution θ.

The interpretation I is inductively defined on terminating terms with respect to $\rhd_{sub} \cup \xrightarrow{R}$, which is well-founded on terminating terms. Moreover, the set $\{I(t') \mid t \xrightarrow{R} t'\}$ is finite because R is finitely branching. Hence, the above definition of I is well-defined. As for argument filterings (Theorem 4.2), this interpretation never destroys well-typedness.

Theorem 5.6 *For any terminating t, $I(t)$ is well-typed and $type(I(t)) = type(t)$.*

Proof. *It can be easily proved by induction on t ordered by $\rhd_{sub} \cup \xrightarrow{R}$.* $\square$

Lemma 5.7 *Let t be a term and θ be a substitution such that $t\theta\!\downarrow$ is terminating. Then, $I(t\theta\!\downarrow) \xrightarrow[C_e]{*} I(t)\theta^I\!\downarrow \xrightarrow[C_e]{*} t\theta^I\!\downarrow$.*

Proof. *We prove the claim by induction on $(\{type(x) \mid x \in dom(\theta)\}, t)$ ordered by the lexicographic combination of the multiset extension $\rhd_s^{mul}$ of $\rhd_s$, and $\rhd_{sub} \cup \xrightarrow{R}$.*

- *In case of $t \equiv \lambda x.t'$: Since $t \rhd_{sub} t'$, we have $I(t'\theta\!\downarrow) \xrightarrow[C_e]{*} I(t')\theta^I\!\downarrow \xrightarrow[C_e]{*} t'\theta^I\!\downarrow$ from the induction hypothesis. Hence we have: $I((\lambda x.t')\theta\!\downarrow) \equiv I(\lambda x.t'\theta\!\downarrow) \equiv \lambda x.I(t'\theta\!\downarrow) \xrightarrow[C_e]{*} \lambda x.I(t')\theta^I\!\downarrow \equiv I(\lambda x.t')\theta^I\!\downarrow$, and $I(\lambda x.t')\theta^I\!\downarrow \equiv \lambda x.I(t')\theta^I\!\downarrow \xrightarrow[C_e]{*} \lambda x.t'\theta^I\!\downarrow \equiv (\lambda x.t')\theta^I\!\downarrow$.*

- *In case of $t \equiv a(\overline{t_n})$ and $a \notin \Delta \cup dom(\theta)$: For each i, since $t \rhd_{sub} t_i$, we have $I(t_i\theta\!\downarrow) \xrightarrow[C_e]{*} I(t_i)\theta^I\!\downarrow \xrightarrow[C_e]{*} t_i\theta^I\!\downarrow$ from the induction hypothesis. Hence we have: $I(a(\overline{t_n})\theta\!\downarrow) \equiv I(a(\overline{t_n\theta\!\downarrow})) \equiv a(\overline{I(t_n\theta\!\downarrow)}) \xrightarrow[C_e]{*} a(\overline{I(t_n)\theta^I\!\downarrow}) \equiv a(\overline{I(t_n)})\theta^I\!\downarrow \equiv I(a(\overline{t_n}))\theta^I\!\downarrow$, and $I(a(\overline{t_n}))\theta^I\!\downarrow \equiv a(\overline{I(t_n)\theta^I\!\downarrow}) \xrightarrow[C_e]{*} a(\overline{t_n\theta^I\!\downarrow}) \equiv a(\overline{t_n})\theta^I\!\downarrow$.*

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- *In case of $t \equiv X \in dom(\theta)$: Obvious from the definition of θ^I.*

- *In case of $t \equiv X(\overline{t_n})$, $X \in dom(\theta)$ and $n > 0$: Thanks to the general assumption $type(X) = type(X\theta)$, we let $X\theta \equiv \lambda \overline{y_n}.a(\overline{u_k})$. Since $type(X) = \alpha_1 \to \cdots \to \alpha_n \to \beta \rhd_s \alpha_i = type(y_i)$ for each i, we have $I(a(\overline{u_k})\{y_i := t_i\theta\downarrow \mid i \in \overline{n}\}\downarrow) \xrightarrow[C_e]{*} I(a(\overline{u_k}))\{y_i := I(t_i\theta\downarrow) \mid i \in \overline{n}\}\downarrow$ from the induction hypothesis. For each i, since $t \rhd_{sub} t_i$, we have $I(t_i\theta\downarrow) \xrightarrow[C_e]{*} I(t_i)\theta^I\downarrow \xrightarrow[C_e]{*} t_i\theta^I\downarrow$ from the induction hypothesis. Hence, by Theorem 3.9 in [21] (if $s \xrightarrow[R]{*} t$ and $\theta \xrightarrow[R]{*} \theta'$ then $s\theta\downarrow \xrightarrow[R]{*} t\theta'\downarrow$), we have: $I(X(\overline{t_n})\theta\downarrow) \equiv I((\lambda\overline{y_n}.a(\overline{u_k}))(\overline{t_n\theta\downarrow})\downarrow) \equiv I(a(\overline{u_k})\{y_i := t_i\theta\downarrow \mid i \in \overline{n}\}\downarrow) \xrightarrow[C_e]{*} I(a(\overline{u_k}))\{y_i := I(t_i\theta\downarrow) \mid i \in \overline{n}\}\downarrow \xrightarrow[C_e]{*} I(a(\overline{u_k}))\{y_i := I(t_i)\theta^I\downarrow \mid i \in \overline{n}\}\downarrow \equiv (\lambda\overline{y_n}.I(a(\overline{u_k})))(\overline{I(t_n)\theta^I\downarrow})\downarrow \equiv X(\overline{I(t_n)})\theta^I\downarrow \equiv I(X(\overline{t_n}))\theta^I\downarrow$, and $I(X(\overline{t_n}))\theta^I\downarrow \equiv X(\overline{I(t_n)})\theta^I\downarrow \equiv (\lambda\overline{y_n}.I(a(\overline{u_k})))(\overline{I(t_n)\theta^I\downarrow})\downarrow \equiv I(a(\overline{u_k}))\{y_i := I(t_i)\theta^I\downarrow \mid i \in \overline{n}\}\downarrow \xrightarrow[C_e]{*} I(a(\overline{u_k}))\{y_i := t_i\theta^I\downarrow \mid i \in \overline{n}\}\downarrow \equiv \lambda\overline{y_n}.I(a(\overline{u_k}))(\overline{t_n\theta^I\downarrow})\downarrow \equiv I(\lambda\overline{y_n}.a(\overline{u_k}))(\overline{t_n\theta^I\downarrow})\downarrow \equiv X(\overline{t_n})\theta^I\downarrow$.*

- *In case of $t \equiv f(\overline{t_n})$ and $f \in \Delta$: For each i, since $t \rhd_{sub} t_i$, we have $I(t_i\theta\downarrow) \xrightarrow[C_e]{*} I(t_i)\theta^I\downarrow \xrightarrow[C_e]{*} t_i\theta^I\downarrow$ from the induction hypothesis. For an arbitrary t'' such that $t \xrightarrow[R\setminus\mathcal{U}(C)]{} t''$, we have $I(t''\theta\downarrow) \xrightarrow[C_e]{*} I(t'')\theta^I\downarrow \xrightarrow[C_e]{*} t''\theta^I\downarrow$ from the induction hypothesis. Hence we have: $I(f(\overline{t_n})\theta\downarrow) \equiv I(f(\overline{t_n\theta\downarrow})) \equiv c(f(\overline{I(t_n\theta\downarrow)}), Red(\{I(t') \mid t\theta\downarrow \to t'\}))\downarrow \xrightarrow[C_e]{*} c(f(\overline{I(t_n\theta\downarrow)}), Red(\{I(t''\theta\downarrow) \mid t \to t''\}))\downarrow \xrightarrow[C_e]{*} c(f(\overline{I(t_n)\theta^I\downarrow}), Red(\{I(t'')\theta^I\downarrow \mid t \to t''\}))\downarrow \equiv c(f(\overline{I(t_n)}), Red(\{I(t'') \mid t \to t''\}))\theta^I\downarrow \equiv I(f(\overline{t_n}))\theta^I\downarrow$, and $I(f(\overline{t_n}))\theta^I\downarrow \equiv c(f(\overline{I(t_n)}), Red(\{I(t'') \mid t \to t''\}))\theta^I\downarrow \equiv c(f(\overline{I(t_n)\theta^I\downarrow}), Red(\{I(t'')\theta^I\downarrow \mid t \to t''\}))\downarrow \xrightarrow[C_e]{} f(\overline{I(t_n)\theta^I\downarrow}) \xrightarrow[C_e]{*} f(\overline{t_n\theta^I\downarrow}) \equiv f(\overline{t_n})\theta^I\downarrow$.* □

For the proof of Theorem 5.3, it is enough to show that $I(t\theta\downarrow) \xrightarrow[C_e]{*} t\theta^I\downarrow$. In fact, the corresponding lemma for STRSs was the claim [27]. However, the proof of the previous lemma required the stronger claim $I(t\theta\downarrow) \xrightarrow[C_e]{*} I(t)\theta^I\downarrow \xrightarrow[C_e]{*} t\theta^I\downarrow$ for applying the induction hypothesis.

Lemma 5.8 *Let t be a term and θ be a permutation such that $t\theta\downarrow$ is terminating. Then, $I(t\theta\downarrow) \equiv I(t)\theta^I\downarrow$.*

Proof. *We prove the claim by induction on t ordered by $\rhd_{sub} \cup \xrightarrow[R]{}$.*

- *In case of $t \equiv \lambda x.t'$: Since $t \rhd_{sub} t'$, we have $I(t'\theta\downarrow) \equiv I(t')\theta^I\downarrow$ from the induction hypothesis. Hence we have: $I((\lambda x.t')\theta\downarrow) \equiv I(\lambda x.t'\theta\downarrow) \equiv \lambda x.I(t'\theta\downarrow) \equiv \lambda x.I(t')\theta^I\downarrow \equiv I(\lambda x.t')\theta^I\downarrow$.*

- *In case of $t \equiv a(\overline{t_n})$ and $a \notin \Delta \cup dom(\theta)$: For each i, since $t \rhd_{sub} t_i$, we have $I(t_i\theta\downarrow) \equiv I(t_i)\theta^I\downarrow$ from the induction hypothesis. Hence we have: $I(a(\overline{t_n})\theta\downarrow) \equiv I(a(\overline{t_n\theta\downarrow})) \equiv a(\overline{I(t_n\theta\downarrow)}) \equiv a(\overline{I(t_n)\theta^I\downarrow}) \equiv I(a(\overline{t_n}))\theta^I\downarrow$.*

- *In case of $t \equiv X(\overline{t_n})$ and $X \in dom(\theta)$: Since θ is a permutation, we let $X\theta\downarrow \equiv X'\downarrow$ for a variable X'. For each i, since $t \rhd_{sub} t_i$, we have $I(t_i\theta\downarrow) \equiv I(t_i)\theta^I\downarrow$ from the induction hypothesis. Hence we have: $I(X(\overline{t_n})\theta\downarrow) \equiv I(X'(\overline{t_n\theta\downarrow})) \equiv X'(\overline{I(t_n\theta\downarrow)}) \equiv X'(\overline{I(t_n)\theta^I\downarrow}) \equiv X(\overline{I(t_n)})\theta^I\downarrow \equiv I(X(\overline{t_n}))\theta^I\downarrow$.*

- *In case of $t \equiv f(\overline{t_n})$ and $f \in \Delta$: For each i, since $t \rhd_{sub} t_i$, we have $I(t_i\theta\downarrow) \equiv I(t_i)\theta^I\downarrow$ from the induction hypothesis. For an arbitrary t'' such that $t \xrightarrow[R\backslash\mathcal{U}(C)]{} t''$, we have $I(t''\theta\downarrow) \equiv I(t'')\theta^I\downarrow$ from the induction hypothesis. Since θ is a permutation, we have $\{ I(t') \mid t\theta\downarrow \to t' \} = \{ I(t''\theta\downarrow) \mid t \to t'' \}$. Hence we have: $I(f(\overline{t_n})\theta\downarrow) \equiv I(f(\overline{t_n\theta\downarrow})) \equiv \mathrm{c}(f(\overline{I(t_n\theta\downarrow)}), Red(\{ I(t') \mid t\theta\downarrow \to t' \})) \equiv \mathrm{c}(f(\overline{I(t_n\theta\downarrow)}),$
$Red(\{ I(t''\theta\downarrow) \mid t \to t'' \})) \equiv \mathrm{c}(f(\overline{I(t_n)\theta^I\downarrow}), Red(\{ I(t'')\theta^I\downarrow \mid t \to t'' \})) \equiv \mathrm{c}(f(\overline{I(t_n)}), Red(\{ I(t'') \mid t \to t'' \}))\theta^I\downarrow \equiv I(f(\overline{t_n}))\theta^I\downarrow$.* ☐

Lemma 5.9 *Let $l \to r \in C \cup \mathcal{U}(C)$ and θ be a substitution such that $r\theta\downarrow$ is terminating. Then, $I(r\theta\downarrow) \equiv r\theta^I\downarrow$.*

Proof. *We show the stronger property $I(t\theta\downarrow) \equiv t\theta^I\downarrow$ for any $l \to r \in C \cup \mathcal{U}(C)$ and $t \in Sub(r)$. We prove the claim by induction on t. Note that we have no case that $t \equiv f(\overline{t_n})$ and $f \in \Delta$.*

- *In case of $t \equiv \lambda x.t'$: Since $t \rhd_{sub} t'$, we have $I(t'\theta\downarrow) \equiv t'\theta^I\downarrow$ from the induction hypothesis. Hence we have: $I((\lambda x.t')\theta\downarrow) \equiv I(\lambda x.t'\theta\downarrow) \equiv \lambda x.I(t'\theta\downarrow) \equiv \lambda x.t'\theta^I\downarrow \equiv (\lambda x.t')\theta^I\downarrow$.*

- *In case of $t \equiv a(\overline{t_n})$ and $a \notin \Delta \cup dom(\theta)$: For each i, since $t \rhd_{sub} t_i$, we have $I(t_i\theta\downarrow) \equiv t_i\theta^I\downarrow$ from the induction hypothesis. Hence we have: $I(a(\overline{t_n})\theta\downarrow) \equiv I(a(\overline{t_n\theta\downarrow})) \equiv a(\overline{I(t_n\theta\downarrow)}) \equiv a(\overline{t_n\theta^I\downarrow}) \equiv a(\overline{t_n})\theta^I\downarrow$.*

- *In case of $t \equiv X(\overline{t_n})$ and $X \in dom(\theta)$: Since $type(X) = type(X\theta)$, we have $X\theta \equiv \lambda\overline{y_n}.a(\overline{u_k})$. For each i, since $t \rhd_{sub} t_i$, we have $I(t_i\theta\downarrow) \equiv t_i\theta^I\downarrow$ from the induction hypothesis. If $t_1, \ldots, t_n$ are mutually distinct bound variables, then $\{y_i := t_i\theta\downarrow \mid i \in \overline{n}\}$ is a permutation, and hence it follows from Lemma 5.8 that $I(X(\overline{t_n})\theta\downarrow) \equiv I((\lambda\overline{y_n}.a(\overline{u_k}))(\overline{t_n\theta})\downarrow) \equiv I(a(\overline{u_k})\{y_i := t_i\theta\downarrow \mid i \in \overline{n}\}\downarrow) \equiv I(a(\overline{u_k}))\{y_i := I(t_i\theta\downarrow) \mid i \in \overline{n}\}\downarrow \equiv I(a(\overline{u_k}))\{y_i := t_i\theta^I\downarrow \mid i \in \overline{n}\}\downarrow \equiv (\lambda\overline{y_n}.I(a(\overline{u_k})))(\overline{t_n\theta^I})\downarrow \equiv I(\lambda\overline{y_n}.a(\overline{u_k}))(\overline{t_n\theta^I})\downarrow \equiv X(\overline{t_n})\theta^I\downarrow$. Otherwise, $I(X(\overline{t_n})\theta\downarrow) \equiv X(\overline{t_n})\theta\downarrow$ and $\theta = \theta^I$, because of $\Delta = \emptyset$.* ☐

Lemma 5.10 *If $s \xrightarrow[R]{} t$ and s is terminating, then $I(s) \xrightarrow[\mathcal{U}(C)\cup C_e]{+} I(t)$.*

Proof. *From $s \xrightarrow[R]{} t$, there exists a rule $l \to r \in R$, a context $E[\,]$, and a substitution θ such that $s \equiv E[l\theta\downarrow]$ and $t \equiv E[r\theta\downarrow]$. We prove the claim by induction on $E[\,]$.*

- *In case of $E[\,] \equiv \square$ and $l \to r \in \mathcal{U}(C)$: From Lemma 5.7 and 5.9, we have: $I(s) \equiv I(l\theta\downarrow) \xrightarrow[C_e]{*} l\theta^I\downarrow \xrightarrow[\mathcal{U}(C)]{} r\theta^I\downarrow \equiv I(r\theta\downarrow) \equiv I(t)$.*

- *In case of $E[\,] \equiv \lambda x.E'[\,]$: $I(\lambda x.E'[l\theta\downarrow]) \equiv \lambda x.I(E'[l\theta\downarrow]) \xrightarrow[\mathcal{U}(C)\cup C_e]{+} \lambda x.I(E'[r\theta\downarrow]) \equiv I(\lambda x.E'[r\theta\downarrow])$.*

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- *In case of $E[] \equiv a(\ldots, E'[], \ldots)$ and $a \notin \Delta$: $I(E[l\theta\downarrow]) \equiv f(.., I(E'[l\theta\downarrow]), ..)$ $\xrightarrow[\mathcal{U}(C) \cup C_e]{+} f(\ldots, I(E'[r\theta\downarrow]), \ldots) \equiv I(E[r\theta\downarrow])$.*

- *In case of $s \equiv f(\overline{s_n})$ and $f \in \Delta$: $I(s) \equiv I(f(\overline{s_n})) \equiv \mathsf{c}(f(\overline{I(s_n)}), Red(\{\, I(v) \mid s \to v \,\})) \xrightarrow[C_e]{} Red(\{\, I(v) \mid s \to v \,\}) \xrightarrow[C_e]{+} I(t)$.* $\qquad\square$

Finally, we give the proof of the main theorem for usable rules:

Proof of Theorem 5.3. Assume that static dependency pairs in C generate an infinite chain $u_0^\sharp \to v_0^\sharp, u_1^\sharp \to v_1^\sharp, \ldots$, in which every $u^\sharp \to v^\sharp \in C$ occurs infinitely many times. Then there exist $\theta_0, \theta_1, \theta_2, \ldots$ such that for each i, $v_i^\sharp \theta_i \downarrow \xrightarrow[R]{*} u_{i+1}^\sharp \theta_{i+1} \downarrow$. Let i be an arbitrary number. From Lemma 5.7, 5.9 and 5.10, we have: $v_i^\sharp \theta_i^I \downarrow \equiv I(v_i^\sharp \theta_i \downarrow) \xrightarrow[\mathcal{U}(C) \cup C_e]{*} I(u_{i+1}^\sharp \theta_{i+1} \downarrow) \xrightarrow[C_e]{*} u_{i+1}^\sharp \theta_{i+1}^I \downarrow$. Hence we have $v_i^\sharp \theta_i^I \downarrow \gtrsim u_{i+1}^\sharp \theta_{i+1}^I \downarrow \gtrsim v_{i+1}^\sharp \theta_{i+1}^I \downarrow$ from $\mathcal{U}(C) \cup C_e \subseteq \gtrsim$. Moreover, from $C \subseteq \gtrsim \cup >$ and $C \cap > \neq \emptyset$, we have $u_j^\sharp \theta_j^I \downarrow > v_j^\sharp \theta_j^I \downarrow$ for infinitely many j. This contradicts the well-foundedness of $>$. $\qquad\square$

6 Conclusion

By using the notion of accessibility [3, 2], we extended in an important way the class of systems to which the static dependency pair method [19] can be applied. We then extended to HRSs some methods initially developed for TRSs: arguments filterings [1] and usable rules [7, 10]. So, together with the subterm criterion for HRSs [19] and the normal higher-order recursive path ordering [12], this paper provides a strong theoretical basis for the development of an efficient automated termination provers for HRSs, since all these methods have been shown quite successful in the termination competition on TRSs [30] and are indeed the basis of current state-of-the-art termination provers for TRSs [8, 10]. We now plan to implement all these techniques, all the more so since some competition on the termination of higher-order rewrite systems is under consideration [24]. Currently, HORPO is the only technique for higher-order rewrite systems that has been implemented [25]. One could also build over [14, 28, 6] to provide certificates for these techniques in the case of HRSs.

However, there are still some theoretical problems. Currently, the static dependency pair method does not handle function definitions involving data type constructors with functional arguments in a satisfactory way like, for instance, the rule $Sum5$ of Van de Pol's formulation of μCRL [31]:

$$\Sigma(\lambda d.Pd) \circ X \to \Sigma((\lambda d.Pd) \circ X)$$

The first reason is that these arguments are not safe (Definition 3.3). This can be fixed by considering a more complex interpretations for base types [2]. The second reason is that it gives rise to the static dependency pair $\Sigma(\lambda d.Pd) \circ X \to Pd \circ X$ the right-hand side of which contains a variable d not occurring in the left-hand side. And, currently, no technique can prove the non-loopingness of this static recursion component, a problem occurring also in [4].

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